

## SMOOTHNESS AND JET SCHEMES

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1

**ABSTRACT.** This paper shows some criteria for a scheme of finite type over an algebraically closed field to be non-singular in terms of jet schemes. For the base field of characteristic zero, the scheme is non-singular if and only if one of the truncation morphisms of its jet schemes is flat. For the positive characteristic case, we obtain a similar characterization under the reducedness condition on the scheme. We also obtain by a simple discussion that the scheme is non-singular if and only if one of its jet schemes is non-singular.

## 1. INTRODUCTION

In 1968 John F. Nash introduced the jet schemes and the arc space of an algebraic and an analytic variety and posed the Nash problem ([7]).

The jet schemes and the arc space are considered to be something to reflect the nature of the singularities of a variety. (The Nash problem itself concerns a connection between the arc space and the singularities.) By looking at the jet schemes over a variety, we can see some properties of the singularities of the variety (see [2], [3], [5], [6]) : for example, if  $X$  is locally a complete intersection variety, the singularities of  $X$  are canonical (resp. terminal) if and only if the jet scheme  $X_m$  is irreducible (resp. normal) for every  $m \in \mathbb{N}$ .

For a non-singular variety  $X$ , the jet schemes are distinguished: the  $m$ -jet scheme  $X_m$  is non-singular for every  $m \in \mathbb{N}$  and every truncation morphism  $\psi_{m',m} : X_{m'} \rightarrow X_m$  is smooth with the fiber  $\mathbb{A}_k^{(m'-m)\dim X}$  for  $m' > m \geq 0$ . Then, it is natural to ask whether these properties characterize the smoothness of the variety  $X$ .

Our results are rather stronger, i.e., only one jet scheme or one truncation morphism is sufficient to characterize the smoothness of the variety  $X$ . In this paper we prove the following:

**Proposition 1.1.** *Let  $k$  be a field of arbitrary characteristic and  $f : X \rightarrow Y$  a morphism of  $k$ -schemes. Then the following are equivalent:*

- (i)  *$f$  is smooth (resp. unramified, étale);*

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- (ii) For every  $m \in \mathbb{N}$ , the morphism  $f_m : X_m \longrightarrow Y_m$  induced from  $f$  is smooth (resp. unramified, étale);
- (iii) There is an integer  $m \in \mathbb{N}$  such that the morphism  $f_m : X_m \longrightarrow Y_m$  is smooth (resp. unramified, étale).

As a corollary of this proposition, we obtain the following:

**Corollary 1.2.** *Let  $k$  be a field of arbitrary characteristic. A scheme  $X$  of finite type over  $k$  is smooth if and only if there is  $m \in \mathbb{Z}_{\geq 0}$  such that  $X_m$  is smooth.*

**Theorem 1.3.** *Let  $k$  be an algebraically closed field of characteristic zero. A scheme  $X$  of finite type over  $k$  is non-singular if and only if there is a pair of integers  $0 \leq m < m'$  such that the truncation morphism  $\psi_{m',m} : X_{m'} \longrightarrow X_m$  is a flat morphism.*

Here, we note that the assumption of the characteristic of the base field in Theorem 1.3 is necessary. We will see a counter example of this statement in positive characteristic (Example 5.3).

If we assume that the scheme  $X$  is reduced, then we have a similar criterion as Theorem 1.3 also for the positive characteristic case.

**Theorem 1.4.** *Let  $k$  be an algebraically closed field of arbitrary characteristic. Assume the scheme  $X$  of finite type over  $k$  is reduced. Then  $X$  is non-singular if and only if there is a pair of integers  $0 < m < m'$  such that the truncation morphism  $\psi_{m',m} : X_{m'} \longrightarrow X_m$  is flat.*

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## 2. PRELIMINARIES ON JET SCHEMES

In this paper, a  $k$ -scheme is always a separated scheme over a field  $k$ .

**Definition 2.1.** Let  $X$  be a scheme of finite type over  $k$  and  $K \supset k$  a field extension. A morphism  $\text{Spec } K[t]/(t^{m+1}) \longrightarrow X$  is called an  $m$ -jet of  $X$ .

**2.2.** Let  $X$  be a scheme of finite type over  $k$ . Let  $\mathcal{S}ch/k$  be the category of  $k$ -schemes and  $\mathcal{S}et$  the category of sets. Define a contravariant functor  $\mathcal{F}_m^X : \mathcal{S}ch/k \longrightarrow \mathcal{S}et$  by

$$\mathcal{F}_m^X(Y) = \text{Hom}_k(Y \times_{\text{Spec } k} \text{Spec } k[t]/(t^{m+1}), X).$$

Then,  $\mathcal{F}_m^X$  is representable by a scheme  $X_m$  of finite type over  $k$ , that is

$$\text{Hom}_k(Y, X_m) \simeq \text{Hom}_k(Y \times_{\text{Spec } k} \text{Spec } k[t]/(t^{m+1}), X).$$

This  $X_m$  is called the *scheme of  $m$ -jets* of  $X$  or the  *$m$ -jet scheme* of  $X$ . For  $m < m'$  the canonical surjection  $k[t]/(t^{m'+1}) \longrightarrow k[t]/(t^{m+1})$  induces a morphism  $\psi_{m',m}^X : X_{m'} \longrightarrow X_m$ , which we call a truncation morphism. In particular, for  $m = 0$   $\psi_{m,0}^X : X_m \longrightarrow X$  is denoted by  $\pi_m^X$ . We denote  $\psi_{m',m}^X$  and  $\pi_m^X$  by  $\psi_{m',m}$  and  $\pi_m$ , respectively, if there is no risk of confusion. By 2.2, a point  $z \in X_m$  gives an  $m$ -jet  $\alpha_z : \text{Spec } K[t]/(t^{m+1}) \longrightarrow X$  and  $\pi_m^X(z) = \alpha_z(0)$ , where  $K$  is the residue field at  $z$  and 0 is the point of  $\text{Spec } K[t]/(t^{m+1})$ . From now on we denote a point  $z$  of  $X_m$  and the corresponding  $m$ -jet  $\alpha_z$  by the common symbol  $\alpha$ .

**2.3.** The canonical inclusion  $k \longrightarrow k[t]/(t^{m+1})$  induces a section  $\sigma_m^X : X \hookrightarrow X_m$  of  $\pi_m^X$ . The image  $\sigma_m^X(x)$  of a point  $x \in X$  is the trivial  $m$ -jet at  $x$  and is denoted by  $x_m$ .

**2.4.** Let  $f : X \longrightarrow Y$  be a morphism of  $k$ -schemes. Then the canonical morphism  $f_m : X_m \longrightarrow Y_m$  is induced for every  $m \in \mathbb{N}$  such that the following diagram is commutative:

$$\begin{array}{ccc} X_m & \xrightarrow{f_m} & Y_m \\ \pi_m^X \downarrow & & \downarrow \pi_m^Y \\ X & \xrightarrow{f} & Y \end{array}.$$

Pointwise, for  $\alpha \in X_m$ ,  $f_m(\alpha)$  is the  $m$ -jet

$$f \circ \alpha : \text{Spec } K[t]/(t^{m+1}) \xrightarrow{\alpha} X \xrightarrow{f} Y.$$

### 3. PROOF OF PROPOSITION 1.1

[*Proof of Proposition 1.1*] (i) $\Rightarrow$ (ii): This implication for smooth and étale cases is already mentioned in [1] and [4]. For the reader's convenience, the proof is included here. Assume for an integer  $m \geq 0$ , a commutative diagram of  $k$ -schemes:

$$\begin{array}{ccc} X_m & \xrightarrow{f_m} & Y_m \\ \uparrow & & \uparrow \\ Z' & \hookrightarrow & Z \end{array}$$

is given, where  $Z' \hookrightarrow Z$  is a closed immersion of affine schemes whose defining ideal is nilpotent. This diagram is equivalent to the following commutative diagram:

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \uparrow & & \uparrow \\ Z' \times \text{Spec } k[t]/(t^{m+1}) & \hookrightarrow & Z \times \text{Spec } k[t]/(t^{m+1}) \end{array}.$$

Here, we note that  $Z' \times \operatorname{Spec} k[t]/(t^{m+1}) \hookrightarrow Z \times \operatorname{Spec} k[t]/(t^{m+1})$  is a closed subscheme with the nilpotent defining ideal. If  $f$  is smooth (resp. unramified, étale), there exists a (resp. there exists at most one, there exists a unique) morphism  $Z \times \operatorname{Spec} k[t]/(t^{m+1}) \rightarrow X$  which makes the two triangles commutative. This is equivalent to the fact that there exists a (resp. there exists at most one, there exists a unique) morphism  $Z \rightarrow X_m$  which makes the two triangles in the first diagram commutative.

(ii) $\Rightarrow$ (iii): trivial.

(iii) $\Rightarrow$ (i): Assume a commutative diagram,

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \varphi \uparrow & & \uparrow \psi \\ Z' & \hookrightarrow & Z \end{array} \quad (1)$$

is given, where  $Z' \hookrightarrow Z$  is a closed immersion of affine schemes whose defining ideal is nilpotent. For an integer  $m \geq 0$ , by composing with the sections  $\sigma_m^X : X \hookrightarrow X_m$ ,  $\sigma_m^Y : Y \hookrightarrow Y_m$ , we obtain the commutative diagram:

$$\begin{array}{ccc} X_m & \xrightarrow{f_m} & Y_m \\ \cup & & \cup \\ X & \xrightarrow{f} & Y \\ \varphi \uparrow & & \uparrow \psi \\ Z' & \hookrightarrow & Z \end{array} \quad (2)$$

Now, if  $f_m$  is smooth (resp. unramified, étale), there exists a (resp. exists at most one, exists a unique) morphism  $Z \rightarrow X_m$  such that the two triangles are commutative in the diagram (2). By composing this morphism  $Z \rightarrow X_m$  with  $\pi_m^X : X_m \rightarrow X$ , we obtain that there exists a (resp. exists at most one, exists a unique) morphism  $Z \rightarrow X$  such that the two triangles in the lower rectangle are commutative.  $\square$

[Proof of Corollary 1.2] In Proposition 1.1, let  $Y = \operatorname{Spec} k$ .  $\square$

#### 4. JET SCHEMES OF A LOCAL ANALYTIC SCHEME

For the proofs of the theorems, here we set up the jet schemes for local analytic schemes. Let  $k$  be an algebraically closed field of arbitrary characteristic. The representability of the following functor follows from [8]. Here, we show the concrete form of the scheme representing the functor.

**Proposition 4.1.** *Let  $\widehat{\mathbb{A}_k^N}$  be the affine scheme  $\operatorname{Spec} \widehat{\mathcal{O}_{\mathbb{A}^N, 0}}$ , where  $\mathcal{O}_{\mathbb{A}^N, 0}$  is the local ring of the origin  $0 \in \mathbb{A}_k^N$  and  $\widehat{\mathcal{O}_{\mathbb{A}^N, 0}}$  is the completion of*

$\mathcal{O}_{\mathbb{A}^N,0}$  at the maximal ideal. Let  $\widehat{\mathcal{F}_m^{\mathbb{A}^N}} : \text{Sch}/k \longrightarrow \text{Set}$  be the functor from the category of  $k$ -schemes to the category of sets defined as follows:

$$\widehat{\mathcal{F}_m^{\mathbb{A}^N}}(Y) := \text{Hom}_k(Y \times_{\text{Spec } k} \text{Spec } k[t]/(t^{m+1}), \widehat{\mathbb{A}_k^N}).$$

For a morphism  $u : Y \longrightarrow Z$  in  $\text{Sch}/k$ ,

$$\widehat{\mathcal{F}_m^{\mathbb{A}^N}}(u) : \text{Hom}_k(Z \times_{\text{Spec } k} \text{Spec } k[t]/(t^{m+1}), \widehat{\mathbb{A}_k^N}) \longrightarrow \text{Hom}_k(Y \times_{\text{Spec } k} \text{Spec } k[t]/(t^{m+1}), \widehat{\mathbb{A}_k^N})$$

is defined by  $f \mapsto f \circ (u \times \text{id})$ .

Then,  $\widehat{\mathcal{F}_m^{\mathbb{A}^N}}$  is representable by the scheme

$$\begin{aligned} (\widehat{\mathbb{A}_k^N})_m &:= \text{Spec } k[[x_{0,1}, x_{0,2}, \dots, x_{0,N}]] [x_{1,1}, \dots, x_{1,N}, \dots, x_{m,1}, \dots, x_{m,N}] \\ &= \text{Spec } k[[\mathbf{x}_0]] [\mathbf{x}_1, \dots, \mathbf{x}_m], \end{aligned}$$

where we denote the multivariables  $(x_{i,1}, x_{i,2}, \dots, x_{i,N})$  by  $\mathbf{x}_i$  for the simplicity of notation.

*Proof.* We may assume that  $Y$  is an affine scheme  $\text{Spec } R$  over  $k$ . Then,

$$\text{Hom}_k(Y \times_{\text{Spec } k} \text{Spec } k[t]/(t^{m+1}), \widehat{\mathbb{A}_k^N}) \simeq \text{Hom}_k(k[[\mathbf{x}_0]], R[t]/(t^{m+1}))$$

Here we have a bijection:

$$\text{Hom}_k(k[[\mathbf{x}_0]], R[t]/(t^{m+1})) \simeq \text{Hom}_k(k[[\mathbf{x}_0]], R) \times R^{mN}$$

by  $\varphi \mapsto (\pi_0 \circ \varphi, \pi_1 \varphi(x_{0,1}), \dots, \pi_1 \varphi(x_{0,N}), \dots, \pi_m \varphi(x_{0,1}), \dots, \pi_m \varphi(x_{0,N}))$ , where,

$$\pi_i : R[t]/(t^{m+1}) \longrightarrow R \quad (i = 0, 1, \dots, m)$$

is the projection of  $R[t]/(t^{m+1}) = R \oplus Rt \oplus \dots \oplus Rt^m \simeq R^{m+1}$  to the  $i$ -th factor. Indeed it gives a bijection, since we have the inverse map

$$\text{Hom}_k(k[[\mathbf{x}_0]], R) \times R^{mN} \longrightarrow \text{Hom}_k(k[[\mathbf{x}_0]], R[t]/(t^{m+1}))$$

by

$$(\varphi_0, a_{1,1}, \dots, a_{1,N}, \dots, a_{m,1}, \dots, a_{m,N}) \mapsto \varphi$$

where  $\varphi \in \text{Hom}_k(k[[\mathbf{x}_0]], R[t]/(t^{m+1}))$  is defined as follows:

For  $\gamma(x_{0,1}, x_{0,2}, \dots, x_{0,N}) \in k[[\mathbf{x}_0]]$ , substituting  $\sum_{i=0}^m x_{i,j} t^i$  into  $x_{0,j}$  ( $j = 1, \dots, N$ ) in  $\gamma$ , we obtain

$$\gamma(\sum \mathbf{x}_i t^i) = \sum_{i=0}^{\infty} \left( \sum_{\sum_{\ell} i_{\ell}=i, 1 \leq j_{\ell} \leq N} \gamma_{i_1, j_1, \dots, i_s, j_s} x_{i_1, j_1} \cdots x_{i_s, j_s} \right) t^i$$

in  $k[[\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_m, t]]$ , where  $\gamma_{i_1, j_1, \dots, i_s, j_s} \in k[[\mathbf{x}_0]]$ . Define  $\varphi(\gamma) \in R[t]/(t^{m+1})$  by

$$\varphi(\gamma) = \sum_{i=0}^m \left( \sum_{\sum_{\ell} i_{\ell} = i, 1 \leq j_{\ell} \leq N} \varphi_0(\gamma_{i_1, j_1, \dots, i_s, j_s}) a_{i_1, j_1} \cdots a_{i_s, j_s} \right) t^i.$$

On the other hand, It is clear that there is a bijection

$$\mathrm{Hom}_k(k[[\mathbf{x}_0]][\mathbf{x}_1, \dots, \mathbf{x}_m], R) \simeq \mathrm{Hom}_k(k[[\mathbf{x}_0]], R) \times R^{mN}$$

by  $\varphi \mapsto (\varphi|_{k[[\mathbf{x}_0]]}, \varphi(x_{1,1}), \dots, \varphi(x_{1,N}), \dots, \varphi(x_{m,1}), \dots, \varphi(x_{m,N}))$ . By this, we have

$$\mathrm{Hom}_k(k[[\mathbf{x}_0]], R[t]/(t^{m+1})) \simeq \mathrm{Hom}_k(k[[\mathbf{x}_0]][\mathbf{x}_1, \dots, \mathbf{x}_m], R),$$

which implies

$$\mathrm{Hom}_k(Y \times \mathrm{Spec} k[t]/(t^{m+1}), \widehat{\mathbb{A}_k^N}) \simeq \mathrm{Hom}_k(Y, \mathrm{Spec} k[[\mathbf{x}_0]][\mathbf{x}_1, \dots, \mathbf{x}_m])$$

This completes the proof.  $\square$

By this proposition, we have the following:

**Corollary 4.2.** *Let  $X \subset \widehat{\mathbb{A}_k^N}$  be a closed subscheme. Let  $I$  be the defining ideal of  $X$  in  $\widehat{\mathbb{A}_k^N}$ . Define a functor  $\mathcal{F}_m^X : \mathrm{Sch}/k \rightarrow \mathrm{Set}$  for this  $X$  in the same way as in the previous proposition.*

*For a power series  $f \in k[[\mathbf{x}_0]]$  we define an element  $F_m \in k[[\mathbf{x}_0]][\mathbf{x}_1, \dots, \mathbf{x}_m]$  as follows:*

$$f\left(\sum_{i=0}^m \mathbf{x}_i t^i\right) = F_0 + F_1 t + F_2 t^2 + \cdots + F_m t^m + \cdots.$$

*Then, the functor  $\mathcal{F}_m^X$  is represented by a scheme  $X_m$  defined in  $(\widehat{\mathbb{A}_k^N})_m = \mathrm{Spec} k[[\mathbf{x}_0]][\mathbf{x}_1, \dots, \mathbf{x}_m]$  by the ideal generated by  $F_i$ 's ( $i \leq m$ ) for all  $f \in I$ . (It is sufficient to take  $F_i$ 's ( $i \leq m$ ) for all generators  $f \in I$ .)*

*Proof.* We use the notation in the proof of the previous proposition. There, we obtained bijections :

$$\begin{aligned} \mathrm{Hom}_k(k[[\mathbf{x}_0]], R[t]/(t^{m+1})) &\stackrel{\Phi}{\simeq} \mathrm{Hom}_k(k[[\mathbf{x}_0]], R) \times R^{mN} \\ &\stackrel{\Psi}{\simeq} \mathrm{Hom}_k(k[[\mathbf{x}_0]][\mathbf{x}_1, \dots, \mathbf{x}_m], R). \end{aligned}$$

Here, for  $Y = \mathrm{Spec} R$ , we have the fact that

$$\mathcal{F}_m^X(Y) = \mathrm{Hom}_k(k[[\mathbf{x}_0]]/I, R[t]/(t^{m+1}))$$

is the subset

$$\{\varphi : k[[\mathbf{x}_0]] \rightarrow R[t]/(t^{m+1}) \mid \varphi(\gamma) = 0 \text{ for generators } \gamma \in I\}$$

of  $\text{Hom}_k(k[[\mathbf{x}_0]], R[t]/(t^{m+1}))$ . The condition  $\varphi(\gamma) = 0$  is equivalent to the conditions  $\pi_i \circ \varphi(\gamma) = 0$  ( $i = 0, 1, \dots, m$ ). Therefore, this subset is mapped by  $\Psi \circ \Phi$  to the subset

$$\left\{ \varphi : k[[\mathbf{x}_0]][\mathbf{x}_1, \dots, \mathbf{x}_m] \longrightarrow R \mid \varphi(x_{i,j}) = a_{i,j}, \text{ for generators } \gamma \in I, \right. \\ \left. \sum_{\ell} \varphi_0(\gamma_{i_1, j_1, \dots, i_s, j_s}) a_{i_1, j_1} \cdots a_{i_s, j_s} = 0 \text{ } (i = 0, 1, \dots, m) \right\}.$$

Let the ideal  $J \subset k[[\mathbf{x}_0]][\mathbf{x}_1, \dots, \mathbf{x}_m]$  be generated by

$$\sum_{\ell} \gamma_{i_1, j_1, \dots, i_s, j_s} x_{i_1, j_1} \cdots x_{i_s, j_s}$$

for generators  $\gamma \in I$ , then it follows that our subset is equal to

$$\text{Hom}_k(k[[\mathbf{x}_0]][\mathbf{x}_1, \dots, \mathbf{x}_m]/J, R).$$

□

**Remark 4.3.** Let  $X \subset \mathbb{A}_k^N$  be a closed subscheme containing the origin 0,  $I_X$  the defining ideal and  $\widehat{X}$  the affine scheme  $\text{Spec } \widehat{\mathcal{O}_{X,0}}$ . Note that the defining ideal  $I$  of  $\widehat{X}$  in  $\widehat{\mathbb{A}_k^N}$  is generated by  $I_X$ . For a polynomial  $f \in k[\mathbf{x}_0]$  we define an element  $F_m \in k[\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_m]$  in the same way as in the previous corollary. Then  $\widehat{X}_m$  is defined in  $(\widehat{\mathbb{A}_k^N})_m = \text{Spec } k[[\mathbf{x}_0]][\mathbf{x}_1, \dots, \mathbf{x}_m]$  by the ideal generated by  $F_i$ 's ( $i \leq m$ ) for generators  $f \in I_X$ .

**Corollary 4.4.** *Under the notation of Remark 4.3, it follows that*

$$\widehat{X}_m = \widehat{X} \times_X X_m.$$

*Proof.* Note that  $F_i \in k[\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_m]$  for a generator  $f$  of  $I_X$  and  $I$  is generated by  $I_X$ . Now the expressions

$$X_m = \text{Spec } k[\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_m]/(F_i)_{f \in I_X}$$

$$\widehat{X}_m = \text{Spec } k[[\mathbf{x}_0]][\mathbf{x}_1, \dots, \mathbf{x}_m]/(F_i)_{f \in I_X}$$

give the required equality. □

**Corollary 4.5.** *Under the notation of Remark 4.3, let  $\pi_m^X$  and  $\pi_m^{\widehat{X}}$  be the canonical projections  $X_m \longrightarrow X$  and  $\widehat{X}_m \longrightarrow \widehat{X}$ , respectively. Then, we obtain the isomorphism of schemes:*

$$(\pi_m^X)^{-1}(0) \simeq (\pi_m^{\widehat{X}})^{-1}(0).$$

**Corollary 4.6.** *Under the notation of Remark 4.3, replacing  $X$  by a sufficiently small neighborhood of 0, we obtain the equivalence that the truncation morphism  $X_{m'} \longrightarrow X_m$  is flat if and only if the truncation morphism  $\widehat{X}_{m'} \longrightarrow \widehat{X}_m$  is flat.*

*Proof.* “Only if” part follows from the base change property for flatness. “If” part follows from the fact that the homomorphism  $\mathcal{O}_{X,0} \longrightarrow \widehat{\mathcal{O}_{X,0}}$  is faithfully flat.  $\square$

**Definition 4.7.** A monomial  $\mathbf{x} = \prod_{\ell=1}^d x_{i_\ell, j_\ell} \in k[[\mathbf{x}_0]][\mathbf{x}_1, \dots, \mathbf{x}_m]$  is called a monomial of *weight*  $w$  if  $w = \sum_{\ell=1}^d i_\ell$ . For an element  $F \in k[[\mathbf{x}_0]][\mathbf{x}_1, \dots, \mathbf{x}_m]$  the order  $\text{ord } F$  is defined as the lowest degree of the monomials in  $\mathbf{x}_0, \dots, \mathbf{x}_m$  that appear in  $F$ .

Note that every monomial in  $F_m$  has weight  $m$  for  $f \in k[[\mathbf{x}_0]]$ .

The next lemma follows from the definition of  $F_m$ :

**Lemma 4.8.** *Let  $f$  be a non-zero power series in  $k[[\mathbf{x}_0]]$  of order  $\geq 1$ .*

- (i) *When  $\text{char } k = 0$ , a monomial  $\prod_{\ell=1}^r x_{0, j_\ell}$  appears in  $f$  if and only if for every  $i_\ell \geq 0$ , the monomial*

$$\prod_{\ell=1}^r x_{i_\ell, j_\ell}$$

*appears in  $F_m$ , where  $\sum_{\ell} i_\ell = m$ .*

*Hence,  $\text{ord } F_m = \text{ord } f$ , and in particular  $F_m \neq 0$  for every  $m$ .*

- (ii) *For any characteristic, a monomial  $\prod_{j=1}^N x_{0, j}^{e_j}$  appears in  $f$  if and only if for every  $i_\ell \geq 0$ , the monomial*

$$\prod_{j=1}^N x_{i_j, j}^{e_j}$$

*appears in  $F_m$ , where  $m = \sum_j e_j i_j$ .*

*Proof.* The statement of “if” part follows immediately from the definition of  $F_m$  for both (i) and (ii). Now assume that  $g = \prod_{\ell=1}^r x_{0, j_\ell}$  is a monomial in  $f$ . By substituting  $\sum_{i \geq 0} x_{i, j} t^i$  into  $x_{0, j}$  in this monomial, we obtain

$$g\left(\sum_{i \geq 0} \mathbf{x}_i t^i\right) = G_0 + G_1 t + G_2 t^2 + \dots$$

Therefore,  $G_m$  is the sum of the monomials of the form  $\prod_{\ell=1}^r x_{i_\ell, j_\ell}$  with  $i_\ell \geq 0$  and  $\sum_{\ell} i_\ell = m$ . If the characteristic of  $k$  is zero, the coefficients of each such monomial is nonzero. And each monomial  $\prod_{\ell=1}^r x_{i_\ell, j_\ell}$  in  $G_m$  is not canceled by the contribution from the other monomials of  $f$ , because the collection  $(j_1, \dots, j_\ell, \dots, j_r)$  assigns the source monomial  $\prod_{\ell=1}^r x_{0, j_\ell}$ . This shows the statement of “only if” part of (i). For the proof of only if part of (ii), let  $g = \prod_j x_{0, j}^{e_j}$  and define  $G_i$  in the same way as in the previous discussion. Then, the monomial  $\prod_j x_{i_j, j}^{e_j}$  appears



with coefficient 1 in  $G_m$  for  $m = \sum_j e_j i_j$ . Therefore, the coefficient of  $\prod_j x_{i_j, j}^{e_j}$  in  $F_m$  is the same as the coefficient of  $\prod_j x_{0, j}^{e_j}$  in  $f$ .  $\square$

**Remark 4.9.** The statement (i) of Lemma 4.8 does not hold for positive characteristic case. For example, let  $p > 0$  be the characteristic of the base field  $k$  and  $f = x_{0,1}^p \in k[[x_{0,1}]]$ . Then  $F_m = x_{i,1}^p$  for  $m = pi$  and  $F_m = 0$  for  $m \not\equiv 0 \pmod{p}$ .

As we saw in the previous section, Corollary 1.2 follows immediately from Proposition 1.1. But here we give another proof of Corollary 1.2 for an algebraically closed base field, since we think that it gives some useful insight into jet schemes.

[*Proof of Corollary 1.2*] We may assume that  $(X, 0) \subset (\widehat{\mathbb{A}_k^N}, 0)$  is a closed subscheme with a singularity at 0, where  $N$  is the embedding dimension of  $(X, 0)$ . Then every element  $f \in I_X$  has order greater than 1. By this, every element  $F_i$  of the defining ideal  $I_{X_m}$  of  $X_m$  in  $(\widehat{\mathbb{A}_k^N})_m$  has order greater than 1. Here, note that  $I_{X_m} \neq 0$ , since  $I_X \neq 0$  and  $F_0 = f$  for  $f \in I_X$ . Therefore the Jacobian matrix of  $I_{X_m}$  is the zero matrix at the trivial  $m$ -jet  $0_m \in X_m$  at 0, which shows that  $0_m$  is a singular point in  $X_m$  for every  $m$ .  $\square$

## 5. PROOFS OF THEOREMS 1.3, 1.4

**5.1.** For the proof of the theorems, we fix the notation as follows: Let  $(X, 0) \subset (\widehat{\mathbb{A}_k^N}, 0)$  be a singularity of embedding dimension  $N$ . Let  $0 \leq m < m'$ ,  $R_m = k[[\mathbf{x}_0]][\mathbf{x}_1, \dots, \mathbf{x}_m]$ ,  $I \subset R_m$  the defining ideal of  $X_m$  in  $(\widehat{\mathbb{A}_k^N})_m$ ,  $R_{m'} = k[[\mathbf{x}_0]][\mathbf{x}_1, \dots, \mathbf{x}_m, \dots, \mathbf{x}_{m'}]$  and  $I' \subset R_{m'}$  the defining ideal of  $X_{m'}$  in  $(\widehat{\mathbb{A}_k^N})_{m'}$ . Let  $M$  be the maximal ideal of  $R_m$  generated by  $\mathbf{x}_0, \dots, \mathbf{x}_m$ .

**Lemma 5.2.** *Under the notation as in 5.1, if there is an element  $F \in I' \cap MR_{m'}$  such that  $F \notin MI' + IR_{m'}$ , then the truncation morphism  $\psi_{m',m} : X_{m'} \rightarrow X_m$  is not flat.*

*Proof.* The truncation morphism  $\psi_{m',m} : X_{m'} \rightarrow X_m$  corresponds to the canonical ring homomorphism  $R_m/I \rightarrow R_{m'}/I'$ . The non-flatness follows from the non-injectivity of the canonical homomorphism:

$$M/I \otimes_{R_m/I} R_{m'}/I' \rightarrow R_{m'}/I'.$$

Since we have an isomorphism of the first module

$$M/I \otimes_{R_m/I} R_{m'}/I' \simeq MR_{m'}/(MI' + IR_{m'}),$$

the existence of an element  $F \in I' \cap MR_{m'}$  such that  $F \notin MI' + IR_{m'}$  gives the non-injectivity.  $\square$

[*Proof of Theorem 1.3*] Assume that the base field  $k$  is algebraically closed and of characteristic zero and  $(X, 0)$  is a singular point of a scheme  $X$  of finite type over  $k$ . Then we will deduce that every truncation morphism  $\psi_{m',m} : X_{m'} \longrightarrow X_m$  ( $m' > m \geq 0$ ) is not flat. For this, it is sufficient to prove that  $\psi_{m',m} : \widehat{X_{m'}} \longrightarrow \widehat{X_m}$  ( $m' > m \geq 0$ ) is not flat by Corollary 4.6. So we may assume that  $X$  is a closed subscheme of  $\widehat{\mathbb{A}_k^N}$  with the embedding dimension  $N$ . Let  $I_X$  be the defining ideal of  $X$  in  $\widehat{\mathbb{A}_k^N}$ . We use the notation of 5.1. Let  $f$  be an element in  $I_X$  with the minimal order  $d$ . Note that  $d \geq 2$ , as  $N$  is the embedding dimension. Then, by Lemma 4.8, (i),  $F_{m+1}$  is not zero and presented as

$$F_{m+1} = g_1(\mathbf{x}_0)x_{m+1,1} + \cdots + g_N(\mathbf{x}_0)x_{m+1,N} + g'(\mathbf{x}_0, \dots, \mathbf{x}_m),$$

where  $\text{ord } F_{m+1} = d$  and some of  $g_i$ 's are not zero. We should note that  $\text{ord } g_i = d - 1$  for all non-zero  $g_i$ 's. As  $\text{ord } g_i \geq 1$ , for every  $i$  and  $\text{ord } g' \geq 1$ , the element  $F_{m+1}$  is in  $MR_{m'}$ . It is clear that  $F_{m+1} \in I'$ . On the other hand, as  $\text{ord } I = \text{ord } I' = d$ , it follows that  $\text{ord } MI' \geq d + 1$  and the initial term of an element  $IR_{m'}$  of order  $d$  is the initial term of an element of  $I$ . Hence, the initial term of an element in  $MI' + IR_{m'}$  of order  $d$  should be the initial term of an element of  $I$ , therefore it should be a polynomial in  $\mathbf{x}_0, \dots, \mathbf{x}_m$ . However, the initial term of  $F_{m+1}$  is not of this form, which implies  $F_{m+1} \notin MI' + IR_{m'}$ . By Lemma 5.2, the non-flatness of  $\psi_{m',m} : X_{m'} \longrightarrow X_m$  follows for every pair  $(m, m')$  with  $0 \leq m < m'$ .  $\square$

**Example 5.3.** The condition  $\text{char } k = 0$  is necessary for Theorem 1.3. Indeed, there are counter examples for Theorem 1.3 in case of positive characteristic. For example, let  $X$  be a scheme defined by  $x_{0,1}^p$  in  $\mathbb{A}_k^1 = \text{Spec } k[x_{0,1}]$  over a field  $k$  of characteristic  $p$ . Let  $r$  be an integer with  $0 < r < p$ . Then, for any positive integer  $q$ , we have

$$X_{pq+r} = \text{Spec } k[x_{0,1}, x_{1,1}, \dots, x_{pq+r,1}] / (x_{0,1}^p, \dots, x_{q,1}^p)$$

and

$$X_{pq} = \text{Spec } k[x_{0,1}, x_{1,1}, \dots, x_{pq,1}] / (x_{0,1}^p, \dots, x_{q,1}^p).$$

It is clear that  $X_{pq+r}$  is flat over  $X_{pq}$ , while  $X$  is singular.

[*Proof of Theorem 1.4*] As in the proof of the previous theorem, we will show the non-flatness of the truncation morphisms, if  $(X, 0)$  is singular. As  $X$  is reduced, some fiber of the truncation morphism  $\psi_{m',m} : X_{m'} \longrightarrow X_m$  has dimension  $\leq (m' - m) \dim(X, 0)$  for a small affine neighborhood  $X$  of 0, if  $\psi_{m',m}$  is flat. (If  $X$  is of equi-dimensional, then

the fiber has dimension  $\dim(X, 0)$ .) Hence, if  $\psi_{m',m}$  is flat, by Corollaries 4.5, 4.6, the dimension of the fiber over a closed point in  $(\pi_m^{\widehat{X}})^{-1}(0)$  by the morphism  $\widehat{\psi_{m',m}} : \widehat{X_{m'}} \longrightarrow \widehat{X_m}$  is  $\leq (m' - m) \dim(X, 0)$ . With remarking this fact and Corollary 4.6, we may assume that  $X$  is a singular closed subscheme of  $\widehat{\mathbb{A}_k^N}$  for the embedding dimension  $N$  of  $(X, 0)$ .

First assume  $m' < d(m+1)$ . Note that for every  $g \in I_X$ ,

$$\overline{G}_i = G_i(\mathbf{0}, \dots, \mathbf{0}, \mathbf{x}_{m+1}, \dots, \mathbf{x}_i) = 0$$

for  $i < d(m+1)$ . This is because every monomial in  $G_i$  has a factor  $x_{\ell,j}$  with  $\ell \leq m$ , since the weight of  $G_i$  is  $i$  ( $< d(m+1)$ ) and  $\text{ord } G_i \geq d$ . Let  $0_m$  be the trivial  $m$ -jet at 0. As  $\psi_{m',m}^{-1}(0_m)$  is defined in  $\mathbb{A}^{(m'-m)N}$  by the ideal generated by  $\overline{G}_i$ 's with  $i \leq m'$  for  $g \in I_X$ , it follows that

$$\psi_{m',m}^{-1}(0_m) \simeq \mathbb{A}^{N(m'-m)},$$

which is a fiber of dimension  $N(m' - m) > (m' - m) \dim(X, 0)$ . Therefore,  $\psi_{m',m}$  is not flat, because otherwise the fiber dimension would be  $(m' - m) \dim(X, 0)$  as we saw before.

Therefore, we may assume that  $m' \geq d(m+1)$ , where  $d = \text{ord } I_X$ . Let  $f \in I_X$  have the order  $d$ . Let  $\prod_j x_{0,j}^{e_j}$  be a monomial with the minimal degree in  $f$ . Then,  $\sum_j e_j = d$  and therefore  $e_j \leq d$  for every  $j$ . Let  $e$  be one of non-zero  $e_j$ 's. By the assumption  $m' \geq d(m+1)$ , there is a positive integer  $i$  such that  $m \leq ie < m'$ . Let  $s$  be minimal among such  $i$ 's. Then  $F_{se} \in I'$  is clear and also we have  $F_{se} \in MR_{m'}$  under the notation of 5.1. Indeed, if a monomial  $\prod_{\ell=1}^u x_{i_\ell, j_\ell}$  of  $F_{se}$  has a factor  $x_{i_\ell, j_\ell}$  with  $i_\ell \geq m+1$ , let this  $i_\ell$  be  $i_1$ . Then  $i_1 \geq m+1 > (s-1)e$ . By this,

$$\sum_{\ell \neq 1} i_\ell < se - (s-1)e = e \leq d \leq u.$$

Therefore, there is at least one  $\ell$  such that  $i_\ell \leq 1 \leq m$ . Hence every monomial of  $F_{se}$  is contained in  $MR_{m'}$ . Now let  $e = e_1$ . As

$$\prod_j x_{0,j}^{e_j}$$

is a monomial of  $f$  of the minimal order  $d$ , by Lemma 4.8,

$$x_{1,s}^e \prod_{j \neq 1} x_{0,j}^{e_j}$$

is a monomial of  $F_{se}$ . Therefore,  $\text{ord } F_{se} = d$ . This monomial does not appear in any element of  $MI' + IR_{m'}$ . Indeed,  $\text{ord } MI' \geq d+1$  and the initial term of an element of  $IR_{m'}$  of order  $d$  must be the initial

term of an element of  $I$ , because of  $\text{ord } I = d$ . Therefore, every initial monomial of an element of  $IR_{m'}$  of order  $d$  is of the form

$$\prod_{\ell} x_{i_{\ell}, j_{\ell}}, \quad \left( \sum_{\ell} i_{\ell} \leq m \right),$$

since  $I$  is generated by  $F_i$ 's with  $i \leq m$  for  $f \in I_X$ . As  $x_{1,s}^e \prod_{j \neq 1} x_{0,j}^{e_j}$  is not of this form, we obtain  $F_{se} \notin IR_{m'} + MI'$ . By this and Lemma 5.2, it follows that  $X_{m'} \longrightarrow X_m$  is not flat for  $m' > m > 0$ .

**Remark 5.4.** In the proof of Theorem 1.4, we used the condition  $m \geq 1$ . It is not clear if the same statement as in the Theorem 1.4 follows for  $m = 0$  in positive characteristic case, i.e., If the base field is of positive characteristic,  $X$  is reduced and  $\pi_{m'} = \psi_{m',0} : X_{m'} \longrightarrow X$  is flat for some  $m' > 0$ , then is  $X$  non-singular?

But in particular, if  $m' = 1$ , it holds true. This is seen as follows: For an affine scheme  $X$  of finite type over  $k$ , the fiber of a point  $x \in X$  by the projection  $\pi_1 : X_1 \longrightarrow X$  is the Zariski tangent space of the point. Therefore  $\dim \pi_1^{-1}(x) = \text{embdim}(X, x)$ . If  $(X, 0)$  is singular and reduced,  $\dim \pi_1^{-1}(x) > \dim(X, 0)$ , while there are points in a small neighborhood of 0 such that the fiber dimension is  $\dim(X, 0)$ . Hence,  $\pi_1$  is not flat.

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